

n -ANGULATED CATEGORIES FROM SELF-INJECTIVE ALGEBRAS

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ABSTRACT. Let \mathcal{C} be a k -linear category with split idempotents, and $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$ an automorphism. We show that there is an n -angulated structure on (\mathcal{C}, Σ) under certain conditions. As an application, we obtain a class of examples of n -angulated categories from self-injective algebras.

1. INTRODUCTION

Let n be an integer greater than or equal to three. Geiss, Keller and Oppermann introduced the notion of n -angulated categories, which is a “higher dimensional” analogue of triangulated categories, and gave the standard construction of n -angulated categories from $(n - 2)$ -cluster tilting subcategories of a triangulated category which are closed under the $(n - 2)$ -nd power of the suspension functor [8]. For $n = 3$, an n -angulated category is nothing but a classical triangulated category. Another examples of n -angulated categories from local algebras were given in [4]. Let R be a commutative local ring with maximal principal ideal $\mathfrak{m} = (p)$ satisfying $\mathfrak{m}^2 = 0$. Then the category of finitely generated free R -modules has a structure of n -angulation whenever n is even, or when n is odd and $2p = 0$ in R . The theory of n -angulated categories has been developed further, we can see [3, 5, 12, 13, 14, 15] for reference. In this note, we devote to provide new class of examples of n -angulated categories.

Throughout this paper let k be an algebraically closed field, and let \mathcal{C} be a k -linear category with split idempotents and $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$ an automorphism. It is natural to ask under which conditions does the category (\mathcal{C}, Σ) has an n -angulation. We first note that if \mathcal{C} is an n -angulated category, then the category $\text{mod}\mathcal{C}$ of contravariant finitely presented and exact functors from \mathcal{C} to $\text{mod}k$ is a Frobenius category. Now we assume that $\text{mod}\mathcal{C}$ is a Frobenius category. Then the stable category $\underline{\text{mod}}\mathcal{C}$ is a triangulated category and the suspension is the cosyzygy functor Ω^{-1} . The automorphism Σ can be extended to an exact functor from $\text{mod}\mathcal{C}$ to $\text{mod}\mathcal{C}$ and thus to a triangle functor of $\underline{\text{mod}}\mathcal{C}$. In this case, (Σ, σ) and $(\Omega^{-n}, (-1)^n)$ are two triangle endofunctors of $\underline{\text{mod}}\mathcal{C}$, where $\sigma : \Sigma\Omega^{-1} \rightarrow \Omega^{-1}\Sigma$ is a natural isomorphism. Heller showed in [11] that there is a bijection between the class of pre-triangulations of (\mathcal{C}, Σ) and the class of isomorphisms of triangle functors from (Σ, σ) to $(\Omega^{-3}, -1)$. Since Heller did not succeed in proving the octahedral axiom, Amiot gave a necessary condition on the functor Σ such that (\mathcal{C}, Σ) has

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a triangulated structure, which is applied to deformed preprojective algebras [1]. In [8] it is showed that Heller's parametrization of pre-triangulations extends to pre- n -angulations. Our first main result is as follows.

Theorem 1.1. *Let \mathcal{C} be a k -linear category with split idempotents and $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$ an automorphism. If $\text{mod}\mathcal{C}$ is a Frobenius category and there exists an exact sequence of exact endofunctors of $\text{mod}\mathcal{C}$*

$$0 \rightarrow \text{Id} \rightarrow X^1 \rightarrow X^2 \rightarrow \cdots \rightarrow X^n \rightarrow \Sigma \rightarrow 0,$$

where all the X^i take values in $\text{proj}\mathcal{C}$. Then (\mathcal{C}, Σ) has an n -angulation structure.

Theorem 1.1 is a higher version of [1, Theorem 8.1]. Since n -angulated categories are more complex than triangulated categories, we should make some technological modifications in the proof.

Let A be a finite-dimensional k -algebra. Given an automorphism σ of A , we denote by ${}_1A_\sigma$ the bimodule structure on A where the action on right is twisted by σ . It is easy to check that ${}_1A_\sigma \otimes_A {}_1A_\tau \cong {}_1A_{\tau\sigma}$, where σ and τ are two automorphisms. A finite-dimensional k -algebra A is said to be *quasi-periodic* if A has a quasi-periodic projective resolution over the enveloping algebra $A^e = A^{\text{op}} \otimes_k A$, i.e., $\Omega_{A^e}^n(A) \cong {}_1A_\sigma$ as A - A -bimodules for some natural number n and some automorphism σ of A . In particular, A is *periodic* if $\Omega_{A^e}^n(A) \cong A$ as bimodules. In this case, if n is minimal, we say A is a periodic algebra of *periodicity* n . It is well known that quasi-periodic algebras are self-injective algebras [10]. Our second main result is as follows.

Theorem 1.2. (*=Theorem 4.3.*) *Let A be a finite-dimensional indecomposable quasi-periodic k -algebra. Assume that $\Omega_{A^e}^n(A) \cong {}_1A_\sigma$ as A - A -bimodules for an automorphism σ of A . Then for each positive integer m , the category $(\text{proj}A, \Sigma)$ has an mn -angulation structure, where Σ is the functor $- \otimes_A A_{\sigma^{-m}} : \text{proj}A \rightarrow \text{proj}A$. In particular, if σ is of finite order l , then $(\text{proj}A, \text{Id}_{\text{proj}A})$ has an ln -angulation structure.*

There are numerous examples of periodic algebras. The most notable examples are preprojective algebras of Dynkin graphs, whose periodicity at most 6. These results have been generalized to deformed preprojective algebras [2]. Dugas showed that each self-injective algebra of finite representation type is periodic [6]. We also can obtain periodic algebras as endomorphism algebras of periodic d -cluster-tilting objects in a triangulated category [7]. Therefore, by Theorem 1.2 we can construct a large class of examples of n -angulated categories from self-injective algebras.

This paper is organized as follows. In Section 2, we recall the definition of n -angulated category and make some preliminaries to prove our first main result. In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorem 1.2 and give some examples.

2. DEFINITIONS AND PRELIMINARIES

Let \mathcal{C} be an additive category equipped with an automorphism $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$. An n - Σ -sequence in \mathcal{C} is a sequence of morphisms

$$X_\bullet = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1).$$

Its *left rotation* is the n - Σ -sequence

$$X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_4 \xrightarrow{f_4} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1 \xrightarrow{(-1)^n \Sigma f_1} \Sigma X_2.$$

We can define *right rotation of an n - Σ -sequence* similarly. An n - Σ -sequence X_\bullet is *exact* if the induced sequence

$$\cdots \rightarrow \mathcal{C}(-, X_1) \rightarrow \mathcal{C}(-, X_2) \rightarrow \cdots \rightarrow \mathcal{C}(-, X_n) \rightarrow \mathcal{C}(-, \Sigma X_1) \rightarrow \cdots$$

is exact. A *morphism of n - Σ -sequences* is a sequence of morphisms $\varphi_\bullet = (\varphi_1, \varphi_2, \dots, \varphi_n)$ such that the following diagram commutes

$$\begin{array}{ccccccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\ \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & & & \downarrow \varphi_n & & \downarrow \Sigma \varphi_1 \\ Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1 \end{array}$$

where each row is an n - Σ -sequence. It is an *isomorphism* if $\varphi_1, \varphi_2, \dots, \varphi_n$ are all isomorphisms in \mathcal{C} .

Definition 2.1. ([8]) An *n -angulated category* is a triple $(\mathcal{C}, \Sigma, \Theta)$, where \mathcal{C} is an additive category, Σ is an automorphism of \mathcal{C} , and Θ is a class of n - Σ -sequences satisfying the following axioms:

- (N1) (a) The class Θ is closed under direct sums and direct summands.
- (b) For each object $X \in \mathcal{C}$ the trivial sequence

$$X \xrightarrow{1_X} X \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow \Sigma X$$

belongs to Θ .

- (c) For each morphism $f_1 : X_1 \rightarrow X_2$ in \mathcal{C} , there exists an n - Σ -sequence in Θ whose first morphism is f_1 .

- (N2) An n - Σ -sequence belongs to Θ if and only if its left rotation belongs to Θ .
- (N3) Each commutative diagram

$$\begin{array}{ccccccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\ \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & & & \downarrow \varphi_n & & \downarrow \Sigma \varphi_1 \\ Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1 \end{array}$$

with rows in Θ can be completed to a morphism of n - Σ -sequences.

- (N4) In the situation of (N3), the morphisms $\varphi_3, \varphi_4, \dots, \varphi_n$ can be chosen such that the mapping cone

$$X_2 \oplus Y_1 \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ \varphi_2 & g_1 \end{pmatrix}} X_3 \oplus Y_2 \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ \varphi_3 & g_2 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -f_n & 0 \\ \varphi_n & g_{n-1} \end{pmatrix}} \Sigma X_1 \oplus Y_n \xrightarrow{\begin{pmatrix} -\Sigma f_1 & 0 \\ \Sigma \varphi_1 & g_n \end{pmatrix}} \Sigma X_2 \oplus \Sigma Y_1$$

belongs to Θ .

Remark 2.2. (a) If $(\mathcal{C}, \Sigma, \Theta)$ is an n -angulated category, then Σ is called a *suspension functor* and Θ is called an *n -angulation* of (\mathcal{C}, Σ) whose elements are called *n -angles*. If Θ only satisfies the three axioms (N1), (N2) and (N3), then Θ is called a *pre- n -angulation* of (\mathcal{C}, Σ) and the triple $(\mathcal{C}, \Sigma, \Theta)$ is called a *pre- n -angulated category*. In this case, an element of Θ is also called an *n -angle*.

(b) An *n - Σ -complex* is a complex $X_\bullet = (X_i, f_i)_{i \in \mathbb{Z}}$ over \mathcal{C} such that $X_{k+n} = \Sigma X_k$ and $f_{k+n} = \Sigma f_k$ for all $k \in \mathbb{Z}$. Let $X_\bullet = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1 \xrightarrow{\Sigma f_1} \Sigma X_2 \xrightarrow{\Sigma f_2} \Sigma X_3 \xrightarrow{\Sigma f_3} \cdots)$

ΣX_1) be an n -angle in a pre- n -angulated category \mathcal{C} , then X_\bullet is exact by [8, Proposition 2.5(a)], which implies that the compositions $f_2 f_1, f_3 f_2, \dots, f_n f_{n-1}, \Sigma f_1 \cdot f_n$ are all zero morphisms. So X_\bullet can be naturally seen as an n - Σ -complex.

(c) The automorphism Σ of \mathcal{C} induces an exact functor from $\text{mod}\mathcal{C}$ to $\text{mod}\mathcal{C}$ defined by $M \mapsto M \cdot \Sigma^{-1}$, which is also denoted by Σ .

Throughout this section, we make the following assumption.

Assumption 2.3. Let \mathcal{C} be a k -linear category with split idempotents and $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$ an automorphism, which satisfying the following two conditions:

- (1) The category $\text{mod}\mathcal{C}$ is a Frobenius category.
- (2) There exists an exact sequence of exact endofunctors of $\text{mod}\mathcal{C}$

$$0 \rightarrow \text{Id} \rightarrow X^1 \rightarrow X^2 \rightarrow \dots \rightarrow X^n \rightarrow \Sigma \rightarrow 0 \quad (2.1)$$

where all the X^i take values in $\text{proj}\mathcal{C}$.

Since \mathcal{C} has split idempotents, the Yoneda functor gives a natural equivalence between \mathcal{C} and $\text{proj}\mathcal{C}$, which is the subcategory of $\text{mod}\mathcal{C}$ consisting of projectives. For convenience we identify \mathcal{C} with $\text{proj}\mathcal{C}$. Since $\text{mod}\mathcal{C}$ is a Frobenius category, we get $\text{proj}\mathcal{C} = \text{inj}\mathcal{C}$ and the quotient category $\underline{\text{mod}}\mathcal{C}$ is a triangulated category with the suspension functor Ω^{-1} . In this case, the automorphism Σ of \mathcal{C} in fact induces a triangle functor of $\underline{\text{mod}}\mathcal{C}$ which is also denoted by Σ . For each $M \in \text{mod}\mathcal{C}$, we fix a short exact sequence $0 \rightarrow M \rightarrow I_M \rightarrow \Omega^{-1}M \rightarrow 0$ with $I_M \in \mathcal{C}$. Thus we obtain a standard injective resolution

$$I_M \rightarrow I_{\Omega^{-1}M} \rightarrow I_{\Omega^{-2}M} \rightarrow \dots \quad (2.2)$$

of M .

Lemma 2.4. *There exists a functorial isomorphism $\alpha : \Sigma \rightarrow \Omega^{-n}$.*

Proof. For each $M \in \text{mod}\mathcal{C}$, by (2.1) and (2.2) we obtain the following commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & M & \longrightarrow & X^1 M & \longrightarrow & X^2 M & \longrightarrow & \dots & \longrightarrow & X^n M & \longrightarrow & \Sigma M & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \alpha_M & & \\ 0 & \longrightarrow & M & \longrightarrow & I_M & \longrightarrow & I_{\Omega^{-1}M} & \longrightarrow & \dots & \longrightarrow & I_{\Omega^{1-n}M} & \longrightarrow & \Omega^{-n}M & \longrightarrow & 0 \end{array}$$

with exact rows, which implies that $\alpha_M : \Sigma M \xrightarrow{\sim} \Omega^{-n}M$ in $\underline{\text{mod}}\mathcal{C}$. For each morphism $f : M \rightarrow M'$ in $\text{mod}\mathcal{C}$, we can easily deduce that $\Omega^{-n}f \cdot \alpha_M = \alpha_{M'} \cdot \Sigma f$ by comparison theorem. \square

Let

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \dots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1 \quad (2.3)$$

be an exact n - Σ -sequence in \mathcal{C} , and $M = \ker f_1$. We note that (2.3) can be seen as the beginning of an injective resolution of M . Since f_n has a factorization $X_n \rightarrow \Sigma M \rightarrow \Sigma X_1$, there exists an isomorphism $\beta_M : \Sigma M \xrightarrow{\sim} \Omega^{-n}M$ in $\underline{\text{mod}}\mathcal{C}$.

Definition 2.5. Denote by Φ the class of exact n - Σ -sequences

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \dots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$$

in \mathcal{C} such that $\beta_{\ker f_1} = \alpha_{\ker f_1}$.

We will show in next section that $(\mathcal{C}, \Sigma, \Phi)$ is an n -angulated category. In the rest of this section, we will give an effective description on the elements in Φ , see Proposition 2.10 for detail.

For each $M \in \text{mod}\mathcal{C}$, we denote by T_M the n - Σ -sequence

$$X^1 M \rightarrow X^2 M \rightarrow \cdots \rightarrow X^n M \rightarrow \Sigma X^1 M$$

induced by the exact sequence (2.1). It is easy to see that $T_M \in \Phi$. We call T_M a *standard n -angle*.

We denote by $C^{\text{ex}}(\text{proj}\mathcal{C})$ the category of acyclic complexes over $\text{proj}\mathcal{C}$. Denote by $C_{n-\Sigma}^{\text{ex}}(\text{proj}\mathcal{C})$ the non-full subcategory of $C^{\text{ex}}(\text{proj}\mathcal{C})$ whose objects are acyclic n - Σ -complexes of the following form

$$(X_\bullet, f_\bullet) = \cdots \rightarrow X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1 \xrightarrow{\Sigma f_1} \Sigma X_2 \rightarrow \cdots,$$

and whose morphisms are n - Σ -periodic. We note that the category $C_{n-\Sigma}^{\text{ex}}(\text{proj}\mathcal{C})$ is a Frobenius category and the projective-injectives are the n - Σ -contractible complexes, i.e., the complexes homotopic to zero with an n - Σ -periodic homotopy. The functor $Z_1 : C_{n-\Sigma}^{\text{ex}}(\text{proj}\mathcal{C}) \rightarrow \text{mod}\mathcal{C}$ which sends a complex (X_\bullet, f_\bullet) to $\ker f_1$ and the functor $T : \text{mod}\mathcal{C} \rightarrow C_{n-\Sigma}^{\text{ex}}(\text{proj}\mathcal{C})$ which sends an object M to T_M are exact functors. Both of the two functors preserve the projective-injectives, thus we get the following lemma.

Lemma 2.6. *The functors Z_1 and T induce triangle functors $Z_1 : K_{n-\Sigma}^{\text{ex}}(\text{proj}\mathcal{C}) \rightarrow \underline{\text{mod}}\mathcal{C}$ and $T : \underline{\text{mod}}\mathcal{C} \rightarrow K_{n-\Sigma}^{\text{ex}}(\text{proj}\mathcal{C})$. Moreover, $Z_1 T = \text{Id}_{\underline{\text{mod}}\mathcal{C}}$.*

An object in $C_{n-\Sigma}^{\text{ex}}(\text{proj}\mathcal{C})$ is simply called an n - Σ -complex in \mathcal{C} . Since an exact n - Σ -sequences in \mathcal{C} can naturally extend to an n - Σ -complex, we can view Φ as a full subcategory of $C_{n-\Sigma}^{\text{ex}}(\text{proj}\mathcal{C})$. In this sense an object in Φ is called a Φ - n - Σ -complex.

Lemma 2.7. *Let (X_\bullet, f_\bullet) be an n - Σ -complex and (Y_\bullet, g_\bullet) a Φ - n - Σ -complex. If $\varphi_\bullet : X_\bullet \rightarrow Y_\bullet$ is homotopy-equivalent, then X_\bullet is also a Φ - n - Σ -complex.*

Proof. Let $M = \ker f_1$ and $N = \ker g_1$, then φ_\bullet induces a morphism $h = Z_1(\varphi_\bullet) : M \rightarrow N$. The morphism h is an isomorphism in $\underline{\text{mod}}\mathcal{C}$ since φ_\bullet is an isomorphism in $K_{n-\Sigma}^{\text{ex}}(\text{proj}\mathcal{C})$. By comparison theorem we have $\Omega^{-n}h \cdot \beta_M = \beta_N \cdot \Sigma h$. We also have $\Omega^{-n}h \cdot \alpha_M = \alpha_N \cdot \Sigma h$ by the naturality of α . Since $\beta_N = \alpha_N$, we obtain that $\beta_M = (\Omega^{-n}h)^{-1} \cdot \alpha_N \cdot \Sigma h = \alpha_M$, so X_\bullet is a Φ - n - Σ -complex. \square

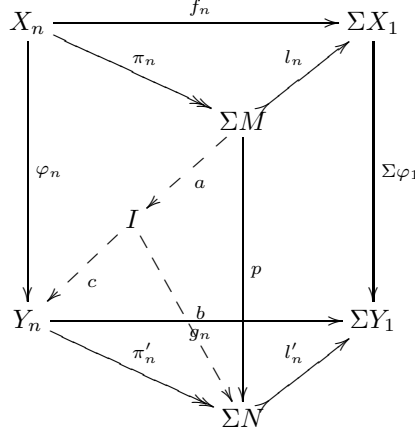
Lemma 2.8. *Each commutative diagram*

$$\begin{array}{ccccccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\ \downarrow \varphi_1 & & \downarrow \varphi_2 & & & & & & & & \downarrow \Sigma \varphi_1 \\ Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1 \end{array}$$

whose rows are Φ - n - Σ -complexes can be extended to an n - Σ -periodic morphism.

Proof. Since the Y_i 's are projective-injectives, by the factorization property of cokernel and the definition of injective we can find morphisms $\varphi_i : X_i \rightarrow Y_i$ such that $\varphi_i f_{i-1} = g_{i-1} \varphi_{i-1}$, where $i = 3, 4, \dots, n$. Let $M = \ker f_1$, $N = \ker g_1$, and $h : M \rightarrow N$ be the morphism induced by the left commutative square. Assume that f_n has a factorization $l_n \pi_n : X_n \twoheadrightarrow \Sigma M \hookrightarrow \Sigma X_1$ and g_n has a factorization $l'_n \pi'_n : Y_n \twoheadrightarrow \Sigma N \hookrightarrow \Sigma Y_1$. The morphism φ_n induces a morphism $p : \Sigma M \rightarrow \Sigma N$

such that $p\pi_n = \pi'_n\varphi_n$. It should be noted that we do not have $\Sigma\varphi_1 \cdot l_n = l'_n p$, but we have $\Sigma\varphi_1 \cdot l_n = l'_n \cdot \Sigma h$.



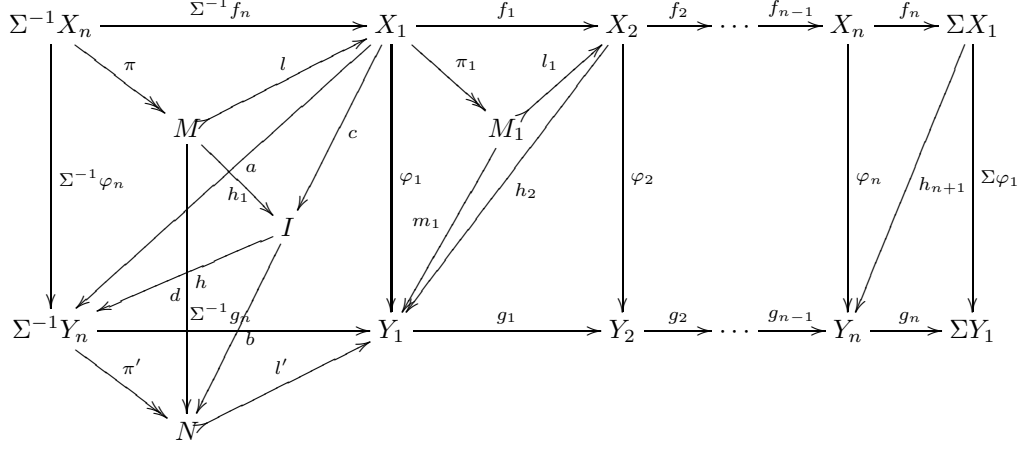
Note that $\beta_N \cdot p = \Omega^{-n}h \cdot \beta_M$ by comparison theorem. On the other hand, we have $\alpha_N \cdot \Sigma h = \Omega^{-n}h \cdot \alpha_M$ by the naturality of α . Since $\beta_M = \alpha_M$ and $\beta_N = \alpha_N$, we obtain that $p = \alpha_N^{-1} \cdot \Omega^{-n}h \cdot \alpha_M = \Sigma h$ in $\underline{\text{mod}}\mathcal{C}$. Thus there exists a projective-injective I in $\underline{\text{mod}}\mathcal{C}$ and morphisms $a : \Sigma M \rightarrow I$ and $b : I \rightarrow \Sigma N$ such that $p - \Sigma h = ba$. As I is projective, there exists a morphism $c : I \rightarrow Y_n$ such that $b = \pi'_n c$. We put $\varphi'_n = \varphi_n - ca\pi_n$, then $\varphi'_n f_{n-1} = \varphi_n f_{n-1} = g_{n-1}\varphi_{n-1}$ and $g_n \varphi'_n = l'_n \pi'_n (\varphi_n - ca\pi_n) = l'_n (p - ba)\pi_n = l'_n \cdot \Sigma h \cdot \pi_n = \Sigma\varphi_1 \cdot l_n \pi_n = \Sigma\varphi_1 \cdot f_n$. Thus $(\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_{n-1}, \varphi'_n)$ is a morphism in $C_{n-\Sigma}^{ex}(\text{proj}\mathcal{C})$. \square

Lemma 2.9. *The functor $Z_1 : K_{n-\Sigma}^{ex}(\text{proj}\mathcal{C}) \rightarrow \underline{\text{mod}}\mathcal{C}$ is full and its kernel is an ideal whose square vanishes. Thus Z_1 detects isomorphisms, that is, if $Z_1(f)$ is an isomorphism in $\underline{\text{mod}}\mathcal{C}$, then f is a homotopy-equivalence.*

Proof. The last assertion follows from the first assertion and [1, Lemma 8.6]. By Lemma 2.6, we have $Z_1 T = \text{Id}_{\underline{\text{mod}}\mathcal{C}}$, which implies that Z_1 is full. We only need to show that $\ker Z_1$ is an ideal whose square vanishes.

Let $\varphi_\bullet : (X_\bullet, f_\bullet) \rightarrow (Y_\bullet, g_\bullet)$ be a morphism of n - Σ -complexes with $Z_1(\varphi_\bullet) = 0$. Let $(M, l : M \rightarrow X_1)$ be the kernel of f_1 and $\Sigma^{-1}f_n = l\pi$. Similarly let $(N, l' : N \rightarrow Y_1)$ be the kernel of g_1 and $\Sigma^{-1}g_n = l'\pi'$. Then $h = Z_1(\varphi_\bullet)$ has a factorization $M \xrightarrow{a} I \xrightarrow{b} N$, where I is projective-injective. Thus there exist two morphisms $c : X_1 \rightarrow I$ and $d : I \rightarrow \Sigma^{-1}Y_n$ such that $a = cl$ and $b = \pi'd$. Let $h_1 = dc$. Note that $(\varphi_1 - \Sigma^{-1}g_n \cdot h_1)\Sigma^{-1}f_n = 0$, there exists a morphism $m_1 : M_1 \rightarrow Y_1$ such that $\varphi_1 - \Sigma^{-1}g_n \cdot h_1 = m_1\pi_1$. Since Y_1 is projective-injective, there exists a morphism $h_2 : X_2 \rightarrow Y_1$ such that $m_1 = h_2l_1$. Thus $\varphi_1 = \Sigma^{-1}g_n \cdot h_1 + h_2f_1$. Similarly we can show that there exist morphisms $h_{i+1} : X_{i+1} \rightarrow Y_i$ such that $\varphi_i = h_{i+1}f_i + g_{i-1}h_i$, $i = 2, 3, \dots, n$. We take $\varphi'_n = (h_{n+1} - \Sigma h_1)f_n$, then $\varphi_n - \varphi'_n = g_{n-1}h_n + \Sigma h_1 \cdot f_n$. Hence the morphism φ_\bullet is homotopy to the morphism $\varphi'_\bullet = (0, 0, \dots, 0, \varphi'_n)$ with

an n - Σ -periodic homotopy (h_1, h_2, \dots, h_n) .



Let $\varphi_\bullet : (X_\bullet, f_\bullet) \rightarrow (Y_\bullet, g_\bullet)$ and $\psi_\bullet : (Y_\bullet, g_\bullet) \rightarrow (Z_\bullet, h_\bullet)$ be morphisms in the kernel of Z_1 . Up to homotopy, we assume that $\varphi_\bullet = (0, 0, \dots, 0, \varphi_n)$ and $\psi_\bullet = (0, 0, \dots, 0, \psi_n)$. Thus we get the following diagram.

$$\begin{array}{ccccccc}
 X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \dots & \xrightarrow{f_{n-2}} & X_{n-2} & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
 \downarrow 0 & & \downarrow 0 & & & & \downarrow 0 & \nearrow a_n & \downarrow \varphi_n & & \downarrow 0 \\
 Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & \dots & \xrightarrow{g_{n-2}} & Y_{n-1} & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1 \\
 \downarrow 0 & & \downarrow 0 & & & & \downarrow 0 & \nearrow \psi_n & \downarrow \psi_n & & \downarrow 0 \\
 Z_1 & \xrightarrow{h_1} & Z_2 & \xrightarrow{h_2} & \dots & \xrightarrow{h_{n-2}} & Z_{n-1} & \xrightarrow{h_{n-1}} & Z_n & \xrightarrow{h_n} & \Sigma Z_1
 \end{array}$$

Since $g_n \varphi_n = 0$ and $\psi_n g_{n-1} = 0$, we have φ_n factors through g_{n-1} and ψ_n factors through g_n . Thus $\psi_n \varphi_n = b_{n+1} g_n g_{n-1} a_n = 0$. So $\psi_\bullet \varphi_\bullet = 0$. \square

The following proposition is a higher version of [1, Proposition 8.7].

Proposition 2.10. *The category of Φ - n - Σ -complexes is equivalent to the category of n - Σ -complexes which are homotopy-equivalent to standard n -angles.*

Proof. Since standard n -angles are Φ - n - Σ -complexes, Lemma 2.7 implies that each n - Σ -complex which is homotopy-equivalent to a standard n -angle is a Φ - n - Σ -complex. Let (X_\bullet, f_\bullet) be a Φ - n - Σ -complex. Let M be the kernel of f_1 . Since $X^1 M$ and $X^2 M$ are projective-injective, we can find morphisms $\varphi_1 : X_1 \rightarrow X^1 M$ and $\varphi_2 : X_2 \rightarrow$

X^2M such that the following diagram is commutative.

$$\begin{array}{ccccccc}
& & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1 \\
& \nearrow & \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \Sigma \varphi_1 \\
M & & X^1 M & \longrightarrow & X^2 M & \longrightarrow & X^3 M \longrightarrow \cdots \longrightarrow X^n M \longrightarrow \Sigma X^1 M \\
& \nwarrow & & & & & \\
M & & & & & &
\end{array}$$

We can complete (φ_1, φ_2) to an n - Σ -periodic morphism $\varphi_\bullet = (\varphi_1, \varphi_2, \dots, \varphi_n)$ from X_\bullet to T_M by Lemma 2.8. Since $Z_1(\varphi_\bullet) = \text{Id}_M$, we obtain that φ_\bullet is a homotopy-equivalence by Lemma 2.9, i.e., X_\bullet is homotopy-equivalent to T_M . \square

3. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. We are going to show that $(\mathcal{C}, \Sigma, \Phi)$ is an n -angulated category.

(N1a) and (N1b) are trivial.

(N1c). Let $f_1 : X_1 \rightarrow X_2$ be a morphism in \mathcal{C} , $A = \ker f_1$ and $B = \text{coker } f_1$. By sequence (2.1) we easily obtain the following commutative diagram

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & A & \xrightarrow{l} & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X^1 B & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & X^{n-3} B & \xrightarrow{\pi_{n-1}} & C & \longrightarrow & 0 \\
& & \parallel & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow g & & \\
0 & \longrightarrow & A & \longrightarrow & I_A & \longrightarrow & I_{\Omega^{-1}A} & \longrightarrow & I_{\Omega^{-2}A} & \longrightarrow & \cdots & \longrightarrow & I_{\Omega^{2-n}A} & \longrightarrow & \Omega^{1-n}A & \longrightarrow & 0
\end{array}$$

with exact rows. Since g is an isomorphism in $\underline{\text{mod}}\mathcal{C}$, we take $h = (\Omega^{-1}g)^{-1}\alpha_A$. Consider the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & C & \xrightarrow{l_{n-1}} & X_n & \xrightarrow{\pi_n} & \Sigma A \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow h \\
0 & \longrightarrow & C & \xrightarrow{i_C} & I_C & \xrightarrow{p_C} & \Omega^{-1}C \longrightarrow 0 \\
& & \downarrow g & & \downarrow & & \downarrow \Omega^{-1}g \\
0 & \longrightarrow & \Omega^{1-n}A & \longrightarrow & I_{\Omega^{1-n}A} & \longrightarrow & \Omega^{-n}A \longrightarrow 0
\end{array}$$

where X_n is the pullback of h and p_C . It is easy to see that

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X^1 B \xrightarrow{f_3} X^2 B \xrightarrow{f_4} \cdots \xrightarrow{f_{n-2}} X^{n-3} B \xrightarrow{l_{n-1}\pi_{n-1}} X_n \xrightarrow{\Sigma l \cdot \pi_n} \Sigma X_1$$

is a Φ - n - Σ -complex.

(N2). Let X_\bullet be a Φ - n - Σ -complex. Since $X_\bullet[1]$ is isomorphic to the left rotation of X_\bullet and $X_\bullet[-1]$ is isomorphic to the right rotation of X_\bullet , we only need to show that $X_\bullet[1]$ and $X_\bullet[-1]$ are Φ - n - Σ -complexes. In fact, X_\bullet is homotopy-equivalent to T_M for some object $M \in \text{mod}\mathcal{C}$ by Proposition 2.10. Thus $X_\bullet[1]$ is homotopy-equivalent to $T_M[1]$. Since $T : \underline{\text{mod}}\mathcal{C} \rightarrow K_{n-\Sigma}^{\text{ex}}(\text{proj}\mathcal{C})$ is a triangle functor, we obtain

that $T_{\Omega^{-1}M}$ is isomorphic to $T_M[1]$. Now $X_\bullet[1]$ is homotopy-equivalent to $T_{\Omega^{-1}M}$, which implies that $X_\bullet[1]$ is a Φ - n - Σ -complex. Similarly we can show $X_\bullet[-1]$ is a Φ - n - Σ -complex.

(N3). It follows from Lemma 2.8.

(N4). Suppose we have a commutative diagram

$$\begin{array}{ccccccccccc} X_\bullet : & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\ & \downarrow \varphi_1 & & \downarrow \varphi_2 & & & & & & & & \downarrow \Sigma \varphi_1 \\ Y_\bullet : & Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1 \end{array}$$

whose rows are Φ - n - Σ -complexes. Let $(M, l : M \rightarrow X_1)$ be the kernel of f_1 , $(N, l' : N \rightarrow Y_1)$ be the kernel of g_1 and $h : M \rightarrow N$ the induced morphism. Then there exist two homotopy-equivalences $a_\bullet : X_\bullet \rightarrow T_M$ and $b_\bullet : T_N \rightarrow Y_\bullet$ by the proof of Proposition 2.10. Let $\phi_\bullet = b_\bullet \cdot T(h) \cdot a_\bullet = (\phi_1, \phi_2, \dots, \phi_n)$ be the morphism from X_\bullet to Y_\bullet . It is easy to see that $(\varphi_1 - \phi_1)l = 0$, so there exists a morphism $h_2 : X_2 \rightarrow Y_1$ such that $\varphi_1 - \phi_1 = h_2 f_1$. Note that $(\varphi_2 - \phi_2 - g_1 h_2) f_1 = 0$, there exists a morphism $h_3 : X_3 \rightarrow Y_2$ such that $\varphi_2 - \phi_2 - g_1 h_2 = h_3 f_2$, i.e., $\varphi_2 - \phi_2 = g_1 h_2 + h_3 f_2$. Let $\varphi_3 = \phi_3 + g_2 h_3$, then $g_3 \varphi_3 = g_3 \phi_3 = \phi_4 f_3$. If we take $\varphi_4 = \phi_4, \dots, \varphi_n = \phi_n$, then $g_i \varphi_i = \varphi_{i+1} f_i$, $i = 4, \dots, n-1$, and $g_n \varphi_n = g_n \phi_n = \Sigma \phi_1 \cdot f_n = \Sigma(\varphi_1 - h_2 f_1) \cdot f_n = \Sigma \varphi_1 \cdot f_n$. Thus $\varphi_\bullet = (\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_n)$ is an n - Σ -periodic morphism and φ_\bullet is n - Σ -homotopic to ϕ_\bullet with the homotopy $(0, h_2, h_3, 0, \dots, 0)$.

It remains to show that the cone $C(\varphi_\bullet)$ is a Φ - n - Σ -complex. In fact, since φ_\bullet is n - Σ -homotopic to $\phi_\bullet = b_\bullet \cdot T(h) \cdot a_\bullet$, where a_\bullet and b_\bullet are homotopy-equivalent, we obtain that the cones $C(\varphi_\bullet)$, $C(\phi_\bullet)$ and $C(T(h))$ are isomorphisms in $K_{n-\Sigma}^{\text{ex}}(\text{proj}\mathcal{C})$. Let $M \xrightarrow{h} N \rightarrow C(h) \rightarrow \Omega^{-1}M$ be a triangle in $\underline{\text{mod}}\mathcal{C}$. Since $T : \underline{\text{mod}}\mathcal{C} \rightarrow K_{n-\Sigma}^{\text{ex}}(\text{proj}\mathcal{C})$ is a triangle functor, $T_M \xrightarrow{T(h)} T_N \rightarrow T_{C(h)} \rightarrow T_M[1]$ is a triangle in $K_{n-\Sigma}^{\text{ex}}(\text{proj}\mathcal{C})$. Thus $C(T(h)) \cong T_{C(h)}$ in $K_{n-\Sigma}^{\text{ex}}(\text{proj}\mathcal{C})$. By these isomorphisms and Proposition 2.10 we get $C(\varphi_\bullet)$ is a Φ - n - Σ -complex. \square

4. APPLICATION TO SELF-INJECTIVE ALGEBRAS

In this section, we will apply Theorem 1.1 to self-injective algebras and give some examples.

Lemma 4.1. ([10, Lemma 1.5]) *Let A be a finite-dimensional indecomposable quasi-periodic k -algebra, then A is a self-injective algebra.*

Lemma 4.2. *Let A be a finite-dimensional self-injective k -algebra. If there exists an exact sequence of A - A -bimodules*

$$0 \rightarrow {}_1A_\sigma \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow A \rightarrow 0 \quad (4.1)$$

where σ is an automorphism of A and the P_i 's are projective as bimodules, then $\text{proj}A$ has an n -angulation structure where the suspension functor is $-\otimes_A A_{\sigma^{-1}}$.

Proof. Since A is self-injective, $\text{mod}A$ is a Frobenius category. If one tensors the sequence (4.1) with ${}_1A_{\sigma^{-1}}$, one obtain the following exact sequence of A - A -bimodules

$$0 \rightarrow A \rightarrow {}_1A_{\sigma^{-1}} \otimes_A P_n \rightarrow {}_1A_{\sigma^{-1}} \otimes_A P_{n-1} \rightarrow \cdots \rightarrow {}_1A_{\sigma^{-1}} \otimes_A P_1 \rightarrow {}_1A_{\sigma^{-1}} \rightarrow 0$$

where all the ${}_1A_{\sigma^{-1}} \otimes_A P_i$ are projective as bimodules. Thus we have the following exact sequence of exact endofunctors of $\text{mod } A$

$$\begin{aligned} 0 \rightarrow \text{Id} \rightarrow - \otimes_A A_{\sigma^{-1}} \otimes_A P_n \rightarrow - \otimes_A A_{\sigma^{-1}} \otimes_A P_{n-1} \rightarrow \cdots \\ \rightarrow - \otimes_A A_{\sigma^{-1}} \otimes_A P_1 \rightarrow - \otimes_A A_{\sigma^{-1}} \rightarrow 0. \end{aligned}$$

Moreover, the functors $- \otimes_A A_{\sigma^{-1}} \otimes_A P_i$ take values in $\text{proj } A$. Note that $e_i A \otimes_A A_{\sigma^{-1}} \cong e_i A_{\sigma^{-1}} \cong \sigma^{-1}(e_i)A$ for each idempotent e_i of A , the functor $- \otimes_A A_{\sigma^{-1}} : \text{proj } A \rightarrow \text{proj } A$ is an automorphism. By Theorem 1.1, we deduce that $(\text{proj } A, - \otimes_A A_{\sigma^{-1}})$ has an n -angulation. \square

Theorem 4.3. *Let A be a finite-dimensional indecomposable quasi-periodic k -algebra. Assume that $\Omega_{A^e}^n(A) \cong {}_1A_\sigma$ as A - A -bimodules for an automorphism σ of A . Then for each positive integer m , the category $(\text{proj } A, \Sigma)$ has an mn -angulation structure, where Σ is the functor $- \otimes_A A_{\sigma^{-m}} : \text{proj } A \rightarrow \text{proj } A$. In particular, if σ is of finite order l , then $(\text{proj } A, \text{Id}_{\text{proj } A})$ has an ln -angulation structure.*

Proof. By Lemma 4.1, we know that A is a self-injective algebra. We claim that there exists an exact sequence of A - A -bimodules

$$0 \rightarrow {}_1A_{\sigma^m} \rightarrow P_{mn} \rightarrow P_{mn-1} \rightarrow \cdots \rightarrow P_{n+1} \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow A \rightarrow 0$$

where the P_i 's are projective as bimodules. Thus the theorem immediately follows from Lemma 4.2. We prove this claim by induction on m . Since $\Omega_{A^e}^n(A) \cong {}_1A_\sigma$ in $\text{mod } A^e$, there exists an exact sequence (4.1), where the P_i 's are projective as A - A -bimodules. Assume now that $m > 1$ and our claim holds for $m - 1$. Applying the functor ${}_1A_{\sigma^{m-1}} \otimes_A -$ to the sequence (4.1), we obtain the following exact sequence of A - A -bimodules

$$0 \rightarrow {}_1A_{\sigma^m} \rightarrow {}_1A_{\sigma^{m-1}} \otimes_A P_n \rightarrow \cdots \rightarrow {}_1A_{\sigma^{m-1}} \otimes_A P_1 \rightarrow {}_1A_{\sigma^{m-1}} \rightarrow 0. \quad (4.2)$$

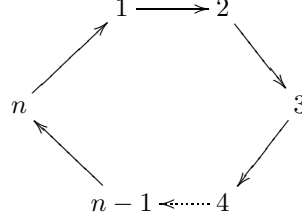
Take $P_{(m-1)n+i} = {}_1A_{\sigma^{m-1}} \otimes_A P_i$, where $i = 1, 2, \dots, n$. Then the $P_{(m-1)n+i}$'s are projective as bimodules. By induction and (4.2), we obtain that our claim holds for each positive integer m . \square

In particular, we get the following easy corollary.

Corollary 4.4. *Let A be a finite-dimensional periodic k -algebra of periodicity n . Then the category $(\text{proj } A, \text{Id}_{\text{proj } A})$ has an n -angulation structure. Moreover, for each positive integer m , the category $(\text{proj } A, \text{Id}_{\text{proj } A})$ has an mn -angulation structure.*

Example 4.5. We will revisit [1, Corollary 9.3]. Let Δ be a graph of generalized Dynkin type and $A = P^f(\Delta)$ be the corresponding deformed preprojective algebra introduced by Bialkowski-Erdmann-Skowroński [2]. We note that if f is zero, then $P^f(\Delta)$ is just the usual preprojective algebra introduced by Gelfand-Ponomarev [9]. By [2, Proposition 3.4], we get $\Omega_{A^e}^3(A) \cong {}_1A_{\sigma^{-1}}$ as A - A -bimodules for an automorphism σ of A of finite order. Moreover, for each idempotent e_i of A , we have $\sigma(e_i) = e_{\nu(i)}$, where ν is the Nakayama permutation. By Theorem 4.1, $\text{proj } A$ is a triangulated category, and the suspension functor $- \otimes_A A_\sigma$ turns out to be the Nakayama functor. Let m be the order of σ , then $(\text{proj } A, \text{Id}_{\text{proj } A})$ has a $3m$ -angulation structure.

Example 4.6. Let $A = kQ_n/I_s$ be a self-injective Nakayama k -algebra, where $n \geq 1$, $s \geq 2$, Q_n is the quiver



and I_s is the ideal generated by paths of length s . It is easy to see that A is of finite representation type. In the notation of Asashiba this is of type $(A_n, \frac{s}{n}, 1)$. By Table 5.2 in [7], we know the periodicity of A is

$$p = \begin{cases} s, & k = 2, n = 1 \text{ and } 2 \nmid s; \\ \frac{2s}{(s, n+1)}, & \text{otherwise.} \end{cases}$$

Thus $(\text{proj}A, \text{Id}_{\text{proj}A})$ has a structure of p -angulated category.

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